

Isomorphisms of some quantum spaces

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Abstract

We consider a series of questions that grew out of determining when two quantum planes are isomorphic. In particular, we consider similar questions for quantum affine spaces and quantum matrix algebras. Additionally, we modify a result by Alev and Dumas to show that two quantum Weyl algebras are isomorphic if and only if their parameters are equal or inverses of each other.

1 Introduction

Quantum rigidity says that automorphism groups of quantum spaces should be small in some sense. Analogously, there should be relatively few isomorphisms between quantum spaces of the same type. In this paper we study the isomorphism problem for quantum affine n -spaces, quantum matrix algebras, and quantum Weyl algebras.

It can be shown that two quantum planes, $\mathcal{O}_p(k^2)$ and $\mathcal{O}_q(k^2)$, are isomorphic if and only if $p = q^{\pm 1}$ (Corollary 3.5). There are multiple approaches to this proof. If one considers only graded isomorphisms, then the result follows by considering $\mathcal{O}_p(k^2)$ and $\mathcal{O}_q(k^2)$ as geometric algebras (see [9]). Our result considers the defining relations of the algebras. As an extension of this problem, we consider isomorphisms of quantum affine space and quantum matrix algebras

Throughout, k is a field and all algebras are k -algebras. Isomorphisms should be read as ‘isomorphisms as k -algebras’. An algebra is said to be *graded* (or \mathbb{N} -graded) if A has a direct sum decomposition $A = \bigoplus_{d \in \mathbb{N}} A_d$ by abelian groups and $A_d A_e \subset A_{d+e}$. An element $a \in A_d$ is said to be *homogeneous* with degree d . If $A_0 = k$, then A is said to be *connected graded*. If A_1 generates A as an algebra, then A is said to be *generated in degree 1* and a basis for A_1 is a *generating basis* for A . If A_1 is finite-dimensional, then A is said to be *affine*. All algebras considered in this paper are affine connected graded and generated in degree 1 with the exception of the quantum Weyl algebras.

If R is an affine connected graded algebra and $a \in R$, then we can decompose a into its homogeneous components, $a = a_0 + \cdots + a_n$, $a_d \in A_d$. If $\Phi : R \rightarrow S$ is a map between affine connected graded algebras and x_i a generating element, we denote by $\Phi_d(x_i)$ the homogeneous degree d component of the image of x_i under Φ . We frequently make use of the graded structure and defining relations of the various algebras. By $T(i, j)$ we mean the image of the defining relation determined by x_i and x_j under Φ . Denote by $T_d(i, j)$ the degree d component of $T(i, j)$.

The definitions presented below are well-known and there are many excellent sources. Our primary source is [3].

Quantum affine space We say $\mathbf{q} \in \mathcal{M}_n(k^\times)$ is multiplicatively antisymmetric if $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for all $i \neq j$. Let $\mathcal{A}_n \subset \mathcal{M}_n(k^\times)$ be the subset of multiplicatively antisymmetric matrices. The symmetric group on n letters, S_n , acts on \mathcal{A}_n by $\sigma.A = [a_{\sigma(i)\sigma(j)}]$ for $A \in \mathcal{A}_n$. We say \mathbf{p} is a permutation of \mathbf{q} if there exists $\sigma \in S_n$ such that $\mathbf{p} = \sigma.\mathbf{q}$.

For $\mathbf{q} \in \mathcal{A}_n$, the quantum affine n -space $\mathcal{O}_{\mathbf{q}}(k^n)$ is defined as the algebra with generating basis $\{x_i\}$, $1 \leq i \leq n$, subject to the relations $x_i x_j = q_{ij} x_j x_i$ for all $1 \leq i, j \leq n$. We prove that two quantum affine spaces, $\mathcal{O}_{\mathbf{p}}(k^n)$ and $\mathcal{O}_{\mathbf{q}}(k^n)$, are isomorphic if and only if \mathbf{p} is a permutation of \mathbf{q} (Theorem 3.4).

Quantum Matrix algebras Let $q \in k^\times$. The single parameter quantum matrix algebras $\mathcal{O}_q(M_n(k))$ is the algebra with generating basis $\{X_{ij}\}$, $1 \leq i, j \leq n$, subject to the relations

$$X_{ij}X_{lm} = \begin{cases} qX_{lm}X_{ij} & i > l, j = m \\ qX_{lm}X_{ij} & i = l, j > m \\ X_{lm}X_{ij} & i > l, j < m \\ X_{lm}X_{ij} + (q - q^{-1})X_{im}X_{lj} & i > l, j > m. \end{cases}$$

The isomorphism result here is identical to that for the quantum planes (Proposition 4.2).

The quantum determinant in $\mathcal{O}_q(M_n(k))$ is

$$D_q = \sum_{\pi \in S_n} (-q)^{\ell(\pi)} X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)},$$

where $\ell(\pi)$ denotes the length of the permutation π . If q is not a root of unity, then the element D_q is central and $\mathcal{Z}(\mathcal{O}_q(M_n(k))) = k[D_q]$. Then quantum coordinate rings of GL_n and SL_n are defined as

$$\begin{aligned} \mathcal{O}_q(\mathrm{GL}_n(k)) &= \mathcal{O}_q(M_n(k))[D_q^{-1}] \text{ and} \\ \mathcal{O}_q(\mathrm{SL}_n(k)) &= \mathcal{O}_q(M_n(k))/(D_q - 1)\mathcal{O}_q(M_n(k)). \end{aligned}$$

If p and q are not roots of unity, then any isomorphism $\mathcal{O}_p(M_n(k)) \rightarrow \mathcal{O}_q(M_n(k))$ preserves the quantum determinant and so extends to isomorphisms $\mathcal{O}_p(\mathrm{GL}_n(k)) \rightarrow \mathcal{O}_q(\mathrm{GL}_n(k))$ and $\mathcal{O}_p(\mathrm{SL}_n(k)) \rightarrow \mathcal{O}_q(\mathrm{SL}_n(k))$. We conjecture that $\mathcal{O}_p(\mathrm{GL}_n(k)) \cong \mathcal{O}_q(\mathrm{GL}_n(k))$ and $\mathcal{O}_p(\mathrm{SL}_n(k)) \cong \mathcal{O}_q(\mathrm{SL}_n(k))$ if and only if $p = q^{\pm 1}$.

The multi-parameter quantum $n \times n$ matrix algebra, $\mathcal{O}_{\lambda, \mathbf{p}}(M_n(k))$, is defined by the generating basis $\{X_{ij}\}$, $1 \leq i, j \leq n$, and parameters $\lambda \in k^\times$, $\mathbf{p} \in \mathcal{A}_n$. The defining relations

are

$$X_{ij}X_{lm} = \begin{cases} p_{il}p_{mj}X_{lm}X_{ij} + (\lambda - 1)p_{il}X_{lj}X_{im} & i > l, j > m \\ \lambda p_{il}p_{mj}X_{lm}X_{ij} & i > l, j \leq m \\ p_{mj}X_{lm}X_{ij} & i = l, j > m. \end{cases}$$

Because of the parameter λ , we do not expect a result as simple as that for the quantum affine spaces. However, we can provide a related result for the case of $n = 2$.

Certain ambiskew polynomial rings In [6], Jordan defines a class of iterated skew polynomial rings with basis $\{x_1, x_2, x_3, x_4\}$ and parameters $a, b, p_1, p_2 \in k^\times$ subject to the relations

$$\begin{aligned} x_4x_1 &= ax_1x_4 & x_2x_1 &= p_1^{-1}a^{-1}x_1x_2 & x_1x_3 &= p_1x_3x_1 \\ x_4x_3 &= bx_3x_4 & x_2x_3 &= p_1b^{-1}x_3x_2 & x_2x_4 &= p_2x_4x_2 + (1 - p_2ab)x_1x_3. \end{aligned}$$

Denote these algebras by $R(a, b, p_1, p_2)$. Making the identifications $x_1 = \lambda q^{-1}X_{12}$, $x_2 = X_{22}$, $x_3 = X_{21}$, $x_4 = X_{11}$, we see that $R(q^{-1}, \lambda^{-1}q, \lambda^{-1}q^2, 1)$ is isomorphic to $\mathcal{O}_{\lambda, q}(M_2(k))$ where $q_{12} = q$. In Section 4.1, we given necessary and sufficient conditions for two rings of the form $R(a, b, p_1, 1)$ to be isomorphic.

Jordan matrix algebra There is an additional ‘quantum matrix algebra’ corresponding to the Jordan plane. As defined in [8], the algebra $\mathcal{O}_J(M_2(k))$ has generating basis $\{a, b, c, d\}$ subject to the relations

$$\begin{aligned} 0 &= [a, b] - b^2 = [b, c] - [a, d] + bd + db = [c, d] - d^2 + ad - bc - bd \\ &= [a, c] - c^2 = [b, c] + [a, d] - cd - dc = [b, d] - d^2 + ad - cb - cd. \end{aligned}$$

The corresponding determinant is given by $D = ad - b(c + d)$. One can check (either by hand or using the Grobner basis package in GAP), that $D^2b = 0$, as asserted in [8]. Hence, $\mathcal{O}_J(M_2(k))$ is not a domain and therefore not isomorphic to the quantum matrix algebras considered above.

Quantum Weyl algebras The quantum Weyl algebra, $A_1^q(k)$, is generated by two elements x and y , subject the relation $xy - qyx = 1$, $q \in k^\times$. It is affine and generated in degree 1 but not graded. Instead, the algebra has a filtration by subspaces $W_d = \{y^i x^j \mid i, j \in \mathbb{N}, i + j \leq d\}$. Then $W_d \subset W_{d+1}$, $W_d W_e \subset W_{d+e}$, and $\bigcup_d W_d = A_1^q(k)$.

We prove that $A_1^p(k) \cong A_1^q(k)$ if and only if $p = q^{\pm 1}$. Our proof of this theorem is split into two propositions (Proposition 5.2 and 5.3). This result was proved recently in greater generality in [10] in the context of quantum generalized Weyl algebras. We offer a different approach, by adapting the proof of Proposition 1.5 in [1] by Alev and Dumas.

2 General results

Our general strategy is to consider the image of certain defining relations under a given isomorphism. The images of the generators can be controlled to a great degree by the graded structure on these algebras.

Throughout this section, let $\Phi : R \rightarrow S$ be an isomorphism between affine connected graded algebras. Let $\{x_i\}$ (resp. $\{y_i\}$) be a generating basis for R (resp. S), $1 \leq i \leq n$.

Lemma 2.1. *The degree 1 component of $\Phi(x_i)$ is nonzero for all i , $1 \leq i \leq n$.*

Proof. The isomorphism Φ is completely determined by its action on the x_i . Hence, the elements $\{\Phi(x_i)\}$ generate all of S . If $\Phi_1(x_i) = 0$ for all i , then $\text{im}\Phi$ contains no degree 1 elements, a contradiction. Suppose there exists l such that $\Phi_1(x_l) = 0$. For convenience, let $l = 1$. Let I be the ideal of S generated by monomials of degree at least 2. If $\overline{\Phi(x_i)}$ is the image of $\Phi(x_i)$ in S/I , then $\text{Span}_k\{\overline{\Phi_1(x_2)}, \dots, \overline{\Phi_1(x_n)}\} = S_1/I$. This gives n linear equations

$$\overline{y_i} = \sum_{j=2}^n \alpha_{ij} \overline{\Phi_1(x_j)}, \quad 1 \leq i \leq n.$$

The graded structure on S implies the y_i are linearly dependent. □

The next step is to show that the constant term of the image of each generator is zero. This will hold so long as the generator is not central.

Lemma 2.2. *If there exists x_j such that $x_i x_j - r x_j x_i = 0$ for some $r \in k^\times$, $r \neq 1$, then $\Phi_0(x_i) = 0$.*

Proof. Suppose $\Phi_0(x_i) \neq 0$ for some i . By hypothesis, there exists x_j and $r \neq 0, 1$ such that $x_i x_j - r x_j x_i = 0$. Let $T(i, j) = \Phi(x_i)\Phi(x_j) - r\Phi(x_j)\Phi(x_i)$. Then

$$T_0(i, j) = \Phi_0(x_i)\Phi_0(x_j)(1 - r) = 0,$$

so $\Phi_0(x_j) = 0$. Thus, $T_1(i, j) = \Phi_0(x_i)\Phi_1(x_j)(1 - r) = 0$. Since $\Phi_1(x_j) \neq 0$ by Lemma 2.1, then $T_1(i, j) \neq 0$, a contradiction. □

Denote by $\mathcal{Z}(A)$ the center of the algebra A . It is clear that the above lemma applies to $\mathcal{O}_q(k^n)$, $\mathcal{O}_q(M_n(k))$, and $\mathcal{O}_{\lambda, q}(M_n(k))$ so long as no generator lies in the center. Hence, we must show that there is some kind of ‘cancellation-type’ result that will allow us to reduce to the case where there are no commutative generators. This is handled in the next lemma.

Lemma 2.3. *If exactly m of the $\{x_i\}$ are central, then exactly m of the $\{y_i\}$ are central.*

Proof. Let $\{z_1, \dots, z_m\}$ (resp. $\{Z_1, \dots, Z_{m'}\}$) be the central generators of R (resp. S). Note that $\{Z_i\}$ may not generate all of $\mathcal{Z}(S)$. However, we do have $\text{Span}_k\{\Phi_1(z_i)\} \subset \text{Span}_k\{Z_i\}$. Hence, $m \leq m'$. Reversing the argument shows $m' \leq m$. □

3 Quantum affine spaces

By ([3], Lemma II.9.7), $\text{GK.dim}(\mathcal{O}_{\mathbf{p}}(k^n)) = n$. Hence, if $\mathcal{O}_{\mathbf{p}}(k^n) \cong \mathcal{O}_{\mathbf{q}}(k^m)$, then $n = m$. Throughout this section, let $\{x_i\}$ (resp. $\{y_i\}$) be a generating basis for $\mathcal{O}_{\mathbf{p}}(k^n)$ (resp. $\mathcal{O}_{\mathbf{q}}(k^n)$).

Lemma 3.1. *If \mathbf{p} is a permutation of \mathbf{q} , then $\mathcal{O}_{\mathbf{p}}(k^n) \cong \mathcal{O}_{\mathbf{q}}(k^n)$.*

Proof. Let $\sigma \in S_n$ be such that $\mathbf{p} = \sigma \cdot \mathbf{q}$. We wish to define a homomorphism $\mathcal{O}_{\mathbf{p}}(k^n) \rightarrow \mathcal{O}_{\mathbf{q}}(k^n)$ via the rule $\Phi(x_i) = y_{\sigma(i)}$. For all i, j , $1 \leq i, j \leq n$, this rule gives

$$\Phi(x_i)\Phi(x_j) - p_{ij}\Phi(x_j)\Phi(x_i) = y_{\sigma(i)}y_{\sigma(j)} - q_{\sigma(i)\sigma(j)}y_{\sigma(j)}y_{\sigma(i)} = 0.$$

Hence, Φ extends to a homomorphism which is clearly bijective. \square

We now proceed to proving the converse.

Lemma 3.2. *Suppose $x_t \notin \mathcal{Z}(\mathcal{O}_{\mathbf{p}}(k^n))$. If $\Phi : \mathcal{O}_{\mathbf{p}}(k^n) \rightarrow \mathcal{O}_{\mathbf{q}}(k^n)$ is an isomorphism, then for all l , $1 \leq l \leq n$, $t \neq l$, there exists $1 \leq r, s \leq n$ with $p_{tl} = q_{rs}$.*

Proof. By Lemma 2.2, $\Phi_0(x_i) = 0$ for each i , $1 \leq i \leq n$. Write $\Phi_1(x_t) = \sum \alpha_i y_i$ and $\Phi_1(x_l) = \sum \beta_i y_i$. Let $T(t, l) = \Phi(x_t)\Phi(x_l) - p_{tl}\Phi(x_l)\Phi(x_t)$. Then

$$0 = T_2(t, l) = (1 - p_{tl}) \left(\sum_{d=1}^n \alpha_d \beta_d y_d^2 \right) + \sum_{1 \leq i \neq j \leq n} (\alpha_i \beta_j - p_{tl} \beta_i \alpha_j) y_i y_j.$$

If $p_{tl} = 1$, then the first sum vanishes and we are left with

$$T_{tl}(2) = \sum_{1 \leq i < j \leq n} [(\alpha_i \beta_j - \beta_i \alpha_j)(1 - q_{ij})] y_i y_j.$$

Hence, either $q_{ij} = 1$ for some $i < j$ or else $\alpha_i \beta_j = \beta_i \alpha_j$ for all $i < j$. In the latter case, there exists m such that $\alpha_m, \beta_m \neq 0$, since otherwise we could choose $i < j$ such that $\alpha_i \beta_j \neq 0$ but $\beta_i \alpha_j = 0$. Then

$$\alpha_m^{-1} \Phi(x_t) = y_m + \sum_{i \neq m} \frac{\alpha_i}{\alpha_m} y_i = y_m + \sum_{i \neq m} \frac{\beta_i}{\beta_m} y_i = \beta_m^{-1} \Phi(x_l).$$

Hence, $\Phi(\alpha_m^{-1} x_t - \beta_m^{-1} x_l) = 0$. Because Φ is an isomorphism, $\alpha_m^{-1} x_t - \beta_m^{-1} x_l = 0$, contradicting the linear independence of x_t and x_l .

Now suppose $p_{tl} \neq 1$. Then $\alpha_d = 0$ or $\beta_d = 0$ for each d . Thus,

$$\begin{aligned} T_2(t, l) &= \sum_{1 \leq i < j \leq n} [(\alpha_i \beta_j - p_{tl} \beta_i \alpha_j) + q_{ij}(\beta_i \alpha_j - p_{tl} \alpha_i \beta_j)] y_i y_j \\ &= \sum_{1 \leq i < j \leq n} [(\alpha_i \beta_j (1 - q_{ij} p_{tl}) + \beta_i \alpha_j (q_{ij} - p_{tl}))] y_i y_j. \end{aligned}$$

By Lemma 2.1, there exists r, s such that $\alpha_r \neq 0$ and $\beta_s \neq 0$. Thus, $p_{tl} = q_{rs}$ if $r < s$ and $p_{tl} = q_{rs}^{-1} = q_{sr}$ if $r > s$. \square

There is one remaining issue. We may have $p_{tl} = q_{rs}$ and $p_{tl} = q_{r's'}$ for distinct pairs (r, s) and (r', s') . This is handled in the next lemma.

Lemma 3.3. *Suppose no generator of $\mathcal{O}_{\mathbf{p}}(k^n)$ is central. If $\Phi : \mathcal{O}_{\mathbf{p}}(k^n) \rightarrow \mathcal{O}_{\mathbf{q}}(k^n)$ is an isomorphism, then \mathbf{p} is a permutation of \mathbf{q} .*

Proof. Write $\Phi_1(x_i) = \sum_{j=1}^n \alpha_{ij} y_j$, $\alpha_{ij} \in k$. Let $M = (\alpha_{ij}) \in \mathcal{M}_n(k)$. Because Φ is an isomorphism, then $\det M \neq 0$.

Suppose there exists j such that $\alpha_{ij}, \alpha_{i'l} \neq 0$ for $i \neq i'$. By Lemma 3.2, for every l there exists s , $1 \leq l, s \leq n$, such that $p_{il} = q_{js} = p_{i'l}$. Thus, we can add linear combinations of rows i and i' in M since the corresponding parameters match.

Therefore, we can reduce M to a matrix M' in which each row contains one nonzero entry. Thus, M' corresponds to an isomorphism $\Psi : \mathcal{O}_{\mathbf{p}}(k^n) \rightarrow \mathcal{O}_{\mathbf{q}}(k^n)$ such that, for each i , $\Psi_1(x_i) = \beta_r y_r$, $\beta_r \in k^\times$. Moreover, the r corresponding to each i is unique. We then define a permutation $\sigma \in S_n$ by $\sigma(i) = r$. Let $T(i, j) = \Psi(x_i)\Psi(x_j) - p_{ij}\Psi(x_j)\Psi(x_i)$. Then

$$0 = T_2(i, j) = \beta_i \beta_j (y_{\sigma(i)} y_{\sigma(j)} - p_{ij} y_{\sigma(j)} y_{\sigma(i)}) = \beta_i \beta_j (q_{\sigma(i)\sigma(j)} - p_{ij}) y_{\sigma(j)} y_{\sigma(i)}.$$

Hence, $q_{\sigma(i)\sigma(j)} = p_{ij}$, so $\mathbf{p} = \sigma \cdot \mathbf{q}$. □

Theorem 3.4. $\mathcal{O}_{\mathbf{p}}(k^n) \cong \mathcal{O}_{\mathbf{q}}(k^n)$ if and only if \mathbf{p} is a permutation of \mathbf{q} .

Proof. Sufficient conditions for an isomorphism are handled in Lemma 3.1.

Suppose $\mathcal{O}_{\mathbf{p}}(k^n) \cong \mathcal{O}_{\mathbf{q}}(k^n)$. If $\mathcal{O}_{\mathbf{p}}(k^n)$ has exactly m commutative generators, then $\mathcal{O}_{\mathbf{q}}(k^n)$ must have exactly the same number by Lemma 2.3. By renaming the generators, there is no loss in assuming $x_1, \dots, x_{n-m} \notin \mathcal{Z}(\mathcal{O}_{\mathbf{p}}(k^n))$ while $x_{n-m+1}, \dots, x_n \in \mathcal{Z}(\mathcal{O}_{\mathbf{p}}(k^n))$ and similarly for the y_i . Hence, $\Phi_1(x_1), \dots, \Phi_1(x_{n-m})$ generate $\text{Span}_k\{y_1, \dots, y_{n-m}\}$. Then Lemma 3.3 gives a permutation $\sigma' \in S_{n-m}$. Thus, we define a permutation, $\sigma \in S_n$ by $\sigma(i) = \sigma'(i)$ if $i \leq n - m$ and otherwise $\sigma(i) = i$. It follows that $\mathbf{p} = \sigma \cdot \mathbf{q}$. □

Corollary 3.5. $\mathcal{O}_p(k^2) \cong \mathcal{O}_q(k^2)$ are isomorphic if and only if $p = q^{\pm 1}$.

4 Quantum matrix algebras

The skew polynomial construction of $\mathcal{O}_q(M_n(k))$ implies $\text{GK.dim}(\mathcal{O}_q(M_n(k))) = n^2$ ([3], Lemma II.9.7). Hence, $\mathcal{O}_p(M_m(k)) \cong \mathcal{O}_q(M_n(k))$ implies $m = n$. Let $\{X_{ij}\}$ (resp. $\{Y_{ij}\}$) be a generating basis for $\mathcal{O}_p(M_n(k))$ (resp. $\mathcal{O}_q(M_n(k))$).

Proposition 4.1. *If $p = q^{\pm 1}$, then $\mathcal{O}_p(M_n(k)) \cong \mathcal{O}_q(M_n(k))$.*

Proof. If $p = q$, then there is nothing to prove. If $p = q^{-1}$, then define a rule $\Phi(X_{ij}) = Y_{rs}$ where $r = (n + 1 - i)$ and $s = (n + 1 - j)$. We must check that this rule satisfies the defining relations for $\mathcal{O}_q(M_n(k))$. We check the first relation and leave the remainder for the reader.

Suppose $i > l$ and $j = m$. Let r, s be as above and let $u = (n + 1 - l)$, $v = n + 1 - m$. Then, $r < u$ and $s = v$. Thus,

$$\Phi(X_{ij})\Phi(X_{lm}) - p\Phi(X_{lm})\Phi(X_{ij}) = Y_{rs}Y_{uv} - pY_{uv}Y_{rs} = (1 - pq)Y_{rs}Y_{uv} = 0.$$

Thus, Φ induces a homomorphism $\mathcal{O}_p(M_n(k)) \rightarrow \mathcal{O}_q(M_n(k))$. This homomorphism is clearly bijective and therefore an isomorphism. \square

We now show that these are the only isomorphisms.

Proposition 4.2. *If $\Phi : \mathcal{O}_p(M_n(k)) \rightarrow \mathcal{O}_q(M_n(k))$ is an isomorphism, then $p = q^{\pm 1}$.*

Proof. Since $\mathcal{O}_1(M_n(k))$ is commutative, then $\mathcal{O}_1(M_n(k)) \cong \mathcal{O}_q(M_n(k))$ implies $q = 1$. Suppose $p, q \neq 1$. Write $\Phi_1(X_{22}) = \sum a_{rs}Y_{rs}$ and $\Phi_1(X_{12}) = \sum b_{rs}Y_{rs}$. Then,

$$\begin{aligned} T_2 &= T((2, 2)(1, 2)) = \Phi(X_{22})\Phi(X_{12}) - p\Phi(X_{12})\Phi(X_{22}) \\ &= (1 - p) \left(\sum_{1 \leq i, j \leq n} a_{ij}b_{ij}Y_{ij}^2 \right) + (1 - p) \sum_{i > l, j > m} (a_{ij}b_{lm} + a_{lm}b_{ij})Y_{lm}Y_{ij} \\ &\quad + \sum_{\substack{1 > l, j = m \\ i = l, j > m}} ((p - q)a_{ij}b_{lm} + (1 - pq)a_{lm}b_{ij})Y_{lm}Y_{ij} \\ &\quad + \sum_{i > l, j < m} ((1 - p)(a_{ij}b_{lm} + a_{lm}b_{ij}) + (q - q^{-1})(a_{lj}b_{im} - pa_{im}b_{lj}))Y_{lm}Y_{ij}. \end{aligned}$$

The coefficients of the Y_{ij}^2 imply that, for all (i, j) , either $a_{ij} = 0$ or $b_{ij} = 0$. By Lemma 2.1, there exists $(i, j) \neq (l, m)$ such that $a_{ij}, b_{lm} \neq 0$. If $i > l$ and $j > m$, then the coefficient of $Y_{lm}Y_{ij}$ is $(1 - p)(a_{ij}b_{lm} + a_{lm}b_{ij}) = 0$. One of $a_{ij}b_{lm}, a_{lm}b_{ij}$ must be zero, which implies that either they are both zero or $(1 - p) = 0$. The latter case contradicts our hypothesis. It then follows that if $i > l$ and $j < m$, then $a_{lj}b_{im} - pa_{im}b_{lj} = 0$. Hence,

$$\begin{aligned} T_2 &= \sum_{\substack{1 > l, j = m \\ i = l, j > m}} ((p - q)a_{ij}b_{lm} + (1 - pq)a_{lm}b_{ij})Y_{lm}Y_{ij} \\ &\quad + (1 - p) \sum_{i > l, j < m} (a_{ij}b_{lm} + a_{lm}b_{ij})Y_{lm}Y_{ij}. \end{aligned}$$

Similar logic to the above shows that $a_{ij}b_{lm} = a_{lm}b_{ij} = 0$ when $i > l$ and $j < m$. Therefore,

$$T_2 = \sum_{\substack{i > l, j = m \\ i = l, j > m}} ((p - q)a_{ij}b_{lm} + (1 - pq)a_{lm}b_{ij})Y_{lm}Y_{ij}.$$

It now follows easily that either $p = q$ or $p = q^{-1}$. \square

4.1 Certain ambiskew polynomial rings

We now consider the ambiskew polynomial rings defined in the introduction. Throughout this section, let $\{x_i\}$ (resp. $\{y_i\}$) be a generating basis for $R(c, d, q_1, q_2)$ (resp. $R(a, b, p_1, p_2)$).

Proposition 4.3. *The following algebras are isomorphic to $R(a, b, p_1, p_2)$:*

- (1) $R(p_1^{-1}a^{-1}, p_1b^{-1}, p_1, p_2^{-1})$,
- (2) $R(b, a, p_1^{-1}, p_2)$,
- (3) $R(p_1^{-1}a^{-1}, p_1b^{-1}, p_1^{-1}, p_2^{-1})$.

Proof. Let $R(c, d, q_1, q_2)$ be one of (1)-(3). We define a rule in each case by

- (1) $x_1 \mapsto aby_1, x_2 \mapsto y_4, x_3 \mapsto y_3, x_4 \mapsto y_2$,
- (2) $x_1 \mapsto p_1y_3, x_2 \mapsto y_2, x_3 \mapsto y_1, x_4 \mapsto y_4$,
- (3) $x_1 \mapsto p_1aby_3, x_2 \mapsto y_4, x_3 \mapsto y_1, x_4 \mapsto y_2$.

We leave it to the reader to verify that these indeed satisfy the defining relations of $R(a, b, p_1, p_2)$ and therefore define bijective homomorphisms $R(c, d, q_1, q_2) \rightarrow R(a, b, p_1, p_2)$. \square

At the present time, we are most interested in the multi-parameter quantum matrix algebras. Hence, we take $p_2, q_2 = 1$. Then there is no confusion in writing $p = p_1$ and $q = q_1$. Moreover, we assume that $c, d, cd, qc, qd^{-1}, qc^2, qd^{-2} \neq 1$. These last two requirements, in terms of the matrix algebras, both translate to $\lambda \neq 1$.

Proposition 4.4. *With the above hypotheses, if $\Phi : R(c, d, q, 1) \rightarrow R(a, b, p, 1)$ is an isomorphism, then $R(c, d, q, 1)$ is one of (1)-(3) in Proposition 4.3.*

Proof. Because of our hypotheses on the parameters, $\Phi_0(x_i) = 0$ for each $i, 1 \leq i \leq 4$. Write, $\Phi_1(x_i) = \sum_{l=1}^n \alpha_{il}y_l$ for each i .

Consider $T_2(13)$. As in the proof of Proposition 4.2, the coefficient of y_j^2 is $\alpha_{1j}\alpha_{3j}(1-q)$ for each $j, 1 \leq j \leq n$. Hence, for each $j, \alpha_{1j} = 0$ or $\alpha_{3j} = 0$. The coefficient of y_4y_2 is $(1-q)(\alpha_{12}\alpha_{34} + \alpha_{14}\alpha_{32})$. Suppose $\alpha_{12} \neq 0$ or $\alpha_{14} \neq 0$, then we must have $\alpha_{32} = 0$ and $\alpha_{34} = 0$. If we now repeat this with the remaining commutation relations for x_1 , then we get $\alpha_{22} = \alpha_{24} = \alpha_{42} = \alpha_{44} = 0$. But then $\dim(\text{Span}\{\Phi_1(x_2), \Phi_1(x_3), \Phi_1(x_4)\}) = 2$, a contradiction. We arrive at the same result if we assume $\alpha_{32} = 0$ or $\alpha_{34} = 0$.

Now $T_2(13) = (\alpha_{11}\alpha_{33}(p-q) + \alpha_{13}\alpha_{31}(1-pq))y_3y_1$ and so we have two possibilities, either $p = q$ or $p = q^{-1}$. This now implies that $\alpha_{21} = \alpha_{23} = \alpha_{41} = \alpha_{43} = 0$.

Case 1 ($p = q$) In this case, $\Phi_1(x_1) = \alpha_{11}y_1$ and $\Phi_1(x_3) = \alpha_{33}y_3$ with $\alpha_{11}, \alpha_{33} \neq 0$. Then

$$\begin{aligned} T_2(41) &= (\alpha_{42}y_2 + \alpha_{44}y_4)\alpha_{11}y_1 - c\alpha_{11}y_1(\alpha_{42}y_2 + \alpha_{44}y_4) \\ &= \alpha_{11}[\alpha_{42}(1-pac)y_2y_1 + \alpha_{44}(a-c)y_1y_4] \end{aligned}$$

If α_{42} and α_{44} are both nonzero, then $1 - pac = 0$ and $a - c = 0$ implying $pc^2 = 1$, contradicting our hypothesis. Hence, either $c = a$ or $c = (pa)^{-1}$, and, depending on the choice, the commutation relation for x_4 and x_3 implies $d = b$ or $d = pb^{-1}$, respectively.

Case 2 ($p = q^{-1}$) In this case, $\Phi_1(x_1) = \alpha_{13}y_3$ and $\Phi_1(x_3) = \alpha_{31}y_1$ with $\alpha_{13}, \alpha_{31} \neq 0$. Then

$$\begin{aligned} T_2(41) &= (\alpha_{42}y_2 + \alpha_{44}y_4)\alpha_{13}y_3 - c\alpha_{13}y_3(\alpha_{42}y_2 + \alpha_{44}y_4) \\ &= \alpha_{13} [\alpha_{42}(pb^{-1} - c)y_3y_2 + \alpha_{44}(b - c)y_3y_4] \end{aligned}$$

If α_{42} and α_{44} are both nonzero, then $pb^{-1} = c$ and $b = c$ implying $qc^2 = 1$, contradicting our hypothesis. Hence, either $c = b$ or $c = pb^{-1}$, and, depending on the choice, the commutation relation for x_4 and x_3 implies $d = a$ or $d = p^{-1}a^{-1}$, respectively. \square

The problem with applying this approach to the general case ($p_2, q_2 \neq 1$) is that $\alpha_{12} \neq 0$ no longer implies $\alpha_{34} = 0$. Further restrictions on the defining parameters would allow this proof to carry through. Otherwise, it seems clear that another approach will be necessary.

5 Quantum Weyl algebras

In this section we assume $\text{char } k = 0$. In this case, the quantum Weyl algebra, $A_1^q(k)$, is simple if and only if $q = 1$. Moreover, $\text{Aut}(A_1^q(k)) \cong k$ unless $q = \pm 1$ [1]. Thus, there is no loss in assuming henceforth that $p, q \neq \pm 1$.

Let $\{X, Y\}$ (resp. $\{x, y\}$) be a generating basis for $A_1^p(k)$ (resp. $A_1^q(k)$) and define the normal elements $Z = XY - YX \in A_1^p(k)$ and $z = xy - yx \in A_1^q(k)$.

Proposition 5.1. *If $p = q^{\pm 1}$, then $A_1^p(k) \cong A_1^q(k)$.*

Proof. If $p = q$, then there is nothing to prove. If $p = q^{-1}$, then define a rule by $\theta(X) = qy$ and $\theta(Y) = -x$. Then,

$$\theta(X)\theta(Y) - q^{-1}\theta(Y)\theta(X) - 1 = -qyx + xy - 1 = 0.$$

Hence, θ extends to a homomorphism $A_1^p(k) \rightarrow A_1^q(k)$. Moreover, the map is bijective and therefore an isomorphism. \square

Recall that $A_1^q(k)$ is PI if and only if q is a primitive root of unity of order ℓ , in which case $\mathcal{Z}(A_1^q(k)) = k[x^\ell, y^\ell]$, and otherwise $\mathcal{Z}(A_1^q(k)) = k$ ([2], Lemma 2.2). Hence, we can split our result into the two following propositions.

Proposition 5.2. *Let $p, q \in k^\times$ with p, q non-roots of unity. If $\theta : A_1^p(k) \rightarrow A_1^q(k)$ is an isomorphism, then $p = q^{\pm 1}$.*

Proof. By [5], the intersection of all nonzero prime ideals in $A_1^p(k)$ (resp. $A_1^q(k)$) is $ZA_1^p(k)$ (resp. $zA_1^q(k)$). Hence, $\theta(ZA_1^p(k)) = \theta(Z)\theta(A_1^p(k)) = \theta(Z)A_1^q(k)$. Since $\theta(Z) \in zA_1^q(k)$, then $\theta(Z) = \lambda z$ for some $\lambda \in A_1^q(k)$. We claim $\lambda \in k^\times$. The ideal $zA_1^q(k)$ is generated by z , so there exists $g \in A_1^q(k)$ such that $g \cdot \lambda z = z$. Hence, λ is a unit in $A_1^q(k)$ and therefore $\lambda \in k^\times$. This gives $\theta(Z) = \lambda z = \lambda(xy - yx) = \lambda(q - 1)yx + \lambda$, and so,

$$\theta(X)\theta(Y) = \theta(Y)\theta(X) + \lambda(q - 1)yx + \lambda.$$

Since θ is an isomorphism,

$$\begin{aligned} 0 &= \theta(XY - pYX - 1) = \theta(X)\theta(Y) - p\theta(Y)\theta(X) - 1 \\ &= (\theta(Y)\theta(X) + \lambda(q - 1)yx + \lambda) - p\theta(Y)\theta(X) - 1 \\ &= (1 - p)\theta(Y)\theta(X) + \lambda(q - 1)yx + (\lambda - 1), \end{aligned}$$

and so,

$$\theta(Y)\theta(X) = (p - 1)^{-1} (\lambda(q - 1)yx + (\lambda - 1)). \quad (\spadesuit)$$

We claim the degrees of $\theta(X)$ and $\theta(Y)$ in $A_1^q(k)$ are both 1. Write $\theta(X) = a = a_0 + \cdots + a_n$, $a_n \neq 0$, and $\theta(Y) = b = b_0 + \cdots + b_m$, $b_m \neq 0$, wherein a_d is the sum of the monomomials of total degree d written according to the filtration $\{y^i x^j \mid i, j \in \mathbb{N}\}$ (and similarly for b_d). Because $A_1^q(k)$ is a domain, the highest degree component of $\theta(Y)\theta(X)$ is $b_m a_n \neq 0$. If n or m is greater than 1, then the left hand side of (\spadesuit) will have degree greater than 2, a contradiction. This proves the claim. Thus, we can write $\theta(X) = \alpha x + \beta y + \gamma$ and $\theta(Y) = \alpha' x + \beta' y + \gamma'$. Substituting this into (\spadesuit) gives

$$\begin{aligned} \alpha' \alpha x^2 + \alpha' \beta xy + \alpha' \gamma x + \beta' \alpha yx + \beta' \beta y^2 + \beta' \gamma y + \gamma' \alpha x + \gamma' \beta y + \gamma' \gamma \\ = \lambda \frac{q - 1}{p - 1} yx + \frac{\lambda - 1}{p - 1}. \end{aligned} \quad (\clubsuit)$$

Thus, $\alpha' \alpha = \beta' \beta = 0$. If $\alpha = \beta = 0$, then $\theta(X)$ is a constant and similarly for $\theta(Y)$ if $\alpha' = \beta' = 0$.

If $\alpha' = \beta = 0$, then (\clubsuit) reduces to

$$\beta' \alpha yx + \beta' \gamma y + \gamma' \alpha x + \gamma' \gamma = (p - 1)^{-1} (\lambda(q - 1)yx + (\lambda - 1)).$$

Thus, $\beta' \alpha \neq 0$ but $\beta' \gamma = \gamma' \alpha = 0$ so $\gamma = \gamma' = 0$. This holds only if $\lambda = 1$ so

$$\begin{aligned} 0 &= \theta(XY - pYX - 1) = \beta' \alpha(xy - pyx) - 1 \\ &= \beta' \alpha(qyx + 1 - pyx) - 1 = \beta' \alpha(q - p)yx + (\beta' \alpha - 1). \end{aligned}$$

Therefore, $p = q$.

Otherwise, $\alpha = \beta' = 0$ and (\clubsuit) reduces to

$$\begin{aligned}\alpha' \beta x y + \alpha' \gamma x + \gamma' \beta y + \gamma' \gamma &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)) \\ \alpha' \beta(qyx + 1) + \alpha' \gamma x + \gamma' \beta y + \gamma' \gamma &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)) \\ q\alpha' \beta y x + \alpha' \gamma x + \gamma' \beta y + (\alpha' \beta + \gamma' \gamma) &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)).\end{aligned}$$

As above, $\gamma = \gamma' = 0$ so

$$\begin{aligned}0 &= \theta(XY - pYX - 1) = \alpha' \beta(yx - pxy) - 1 \\ &= \alpha' \beta(yx - p(qyx + 1)) - 1 = \alpha' \beta(1 - pq)yx - (p\alpha' \beta + 1).\end{aligned}$$

Therefore, $p = q^{-1}$. \square

Proposition 5.3. *Let $p, q \in k^\times$ with $p, q \neq \pm 1$ primitive roots of unity. If $\theta : A_1^p(k) \rightarrow A_1^q(k)$ is an isomorphism, then $p = q^{\pm 1}$.*

Proof. Decompose $\theta(X)$, $\theta(Y)$, a_n and b_m as before. Again, choose r, s minimal such that $a_{n,r}, b_{m,s} \neq 0$. As $0 = \theta(XY - pYX - 1) = ab - pba - 1$, the highest y -degree term in $a_n b_m - p b_m a_n$ is $a_{n,r} b_{m,s} [q^{r(m-s)} - p q^{s(n-r)}] y^{n+m-r-s} x^{r+s} = 0$. Hence,

$$q^{r(m-s)} - p q^{s(n-r)} = q^{r(m-s)}(1 - p q^{ns-mr}) = 0.$$

This implies that $p = q^{mr-nr}$. Likewise, $q = p^t$ for some $t \in \mathbb{N}$. Thus, p and q are roots of unity of the same order ℓ . Hence, $\mathcal{Z}(A_1^p(k)) = k[X^\ell, Y^\ell]$ and $\mathcal{Z}(A_1^q(k)) = k[x^\ell, y^\ell]$. Then $\theta(X^\ell) = a^\ell = a'_{n\ell} + a'_{n\ell-1} + \cdots a'_0$ where

$$a'_{n\ell} = \alpha_{n,r}^\ell q^\nu y^{n-r} x^{r\ell} + \sum_{j=0}^{r\ell-1} \alpha'_{n\ell,j} y^{n\ell-j} x^j, \quad (1)$$

with $\nu \in \mathbb{Z}$ and $\alpha'_{n\ell,j} \in k$. Similarly, $\theta(Y^\ell) = b^\ell = b'_{m\ell} + b'_{m\ell-1} + \cdots b'_0$ where

$$b'_{m\ell} = \beta_{m,s}^\ell q^w y^{m-s} x^{s\ell} + \sum_{j=0}^{s\ell-1} \beta'_{m\ell,j} y^{m\ell-j} x^j. \quad (2)$$

The restriction of θ to the centers of the respective algebras determines an automorphism of the polynomial ring in two variables. Because X^ℓ and Y^ℓ are central, then $\alpha'_e = \beta'_e = 0$ if $e \not\equiv 0$ modulo ℓ and $\alpha'_{n\ell,j} = \beta'_{m\ell,j} = 0$ if $j \not\equiv 0$ modulo ℓ . Lemma 2 of [7] shows that there are three possibilities for an automorphism of the polynomial ring in two variables (see also [1]).

Case 1: There exists $t \in \mathbb{Z}_{>0}$ and $\lambda \in k$ such that $a'_{n\ell} = \lambda(b'_{m\ell})^t$. Substituting into (1) and (2) shows that $r = st$ and $n = mt$, so $ns = mr$. Then $p q^{ns-mr} = 1$ implies $p = 1$, a contradiction.

Case 2: There exists $t \in \mathbb{Z}_{>0}$ and $\lambda \in k$ such that $b'_{m\ell} = \lambda(a'_{n\ell})^t$. This gives the same contradiction as above.

Case 3: $\theta(X^\ell) = \zeta x^\ell + \xi y^\ell + \omega$ and $\theta(Y^\ell) = \zeta' x^\ell + \xi' y^\ell + \omega'$ with $\zeta, \xi, \omega, \zeta', \xi', \omega' \in k$. Hence, the degree of $\theta(X)$ and $\theta(Y)$ is 1 and we refer to the proof of Proposition 5.2. \square

Acknowledgements

The author would like to thank his advisor, Allen Bell, for his careful edits and patient guidance throughout this project. Also, thanks to the organizers of the 31st Ohio State-Denison Conference where parts of this were first presented.

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